Tutorial 11

In the problems below, V is a finite-dimensional vector space and $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ unless otherwise stated.

1. Let $T \in \mathcal{L}(V)$. Show there exists $S \in \mathcal{L}(V)$ and $c \in \mathbb{F}$ such that trace $S = 0$ and
$$T = S + cI$$
2. Suppose $\mathbb{F} = \mathbb{R}$ and define
$$A = \begin{bmatrix} 4 & c & c^7 \\ c & 0 & \sin(c) \\ 0 & 0 & 4 \end{bmatrix}$$
where $c \in \mathbb{R}$. Find all values of c such that $A = B^2$ for some real matrix B.
3. Suppose $\mathbb{F} = \mathbb{C}$ and define
$$A = \begin{bmatrix} c & -2 & -c \\ 1 & c^2 - 3 & c^5 - 2 \\ \cos(\pi c) & 2 & 4 - 3c \end{bmatrix}$$
where $c \in \mathbb{C}$. Find all values of c such that A is nilpotent.
4. Suppose V is an inner product space. Let $T \in \mathcal{L}(V)$ be such that
$$T^* = -T$$
We say such an operator is *skew-adjoint*.
   (a) Show that if $\mathbb{F} = \mathbb{R}$ and $\dim V$ is odd then T is not invertible.
   (b) Show that if $\mathbb{F} = \mathbb{C}$, $\dim V$ is odd, and T is invertible then T has a nonreal eigenvalue.
5. Note that 247, 418, and 931 are all integer multiples of 19. **Without actually calculating the determinant**, prove that $\det A$ is also an integer multiple of 19, where
$$A = \begin{pmatrix} 2 & 4 & 7 \\ 4 & 1 & 8 \\ 9 & 3 & 1 \end{pmatrix}$$

6. Suppose $\mathbb{F} = \mathbb{C}$ and let $T \in \mathcal{L}(V)$ be invertible such that all eigenvalues of T are real. Further suppose

$$\text{trace } T^2 = \text{trace } T^3 = \text{trace } T^4$$

What is trace T?

7. Let $f\colon \mathcal{L}(V) \to \mathbb{F}$ be a linear functional such that for all $S, T \in \mathcal{L}(V)$,

$$f(ST) = f(TS)$$

and $f(I) = \dim V$. Show that f is the trace map.